An Accelerated Algebraic Reconstruction Technique based on the Newton-Raphson Scheme

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***Abstract*—The idea presented here is based on the Newton-Raphson root-finding methodology for localizing the minimum of a function. The proposed algorithm follows the iterative approach of the traditional Algebraic Reconstruction Technique (ART) with the introduction of a new correction method, similar to the Newton-Raphson scheme generalized to several dimensions. The definition of the derivative in this method causes an acceleration in the convergence speed, which results to a respectable drop of the number of iterations needed to minimize the quadratic deviation. The major issue was the definition of a *Cost Function* and its first and second derivative, the equivalent root of which would lead to the detection of the local minimum. This *Cost Function* contains the squared difference of the measured and the reconstructed projections in the appropriate matrix notation and takes into account the derivatives with respect to neighborhood rays and projection angles. Apart from the formalism, the quality of the proposed reconstruction and its convergence speed with respect to the traditional ART is discussed in this work.**

I. Introduction

The algebraic approach to image reconstruction from projections consists basically of two iterative techniques: the Algebraic Reconstruction Technique (ART) [1] and the Simultaneous Iterative Reconstruction Technique (SIRT) [2]. ART-type methods are sequential in nature; they implement a correction to the estimated image vector in such a way that the updated estimate will satisfy a single ray-sum equation representing a ray integral. SIRT-type methods are quadratic optimization methods; in their approach they attempt to correct for errors in all ray-sum equations simultaneously. The greatest advantage of ART in the computed tomography is the ability to produce better images than other methods from fewer projections.

The original ART algorithm (sometimes referred to as Additive ART) is the basis of many variants, developed to improve various aspects not addressed by the original ART. A simultaneous application of the error correction terms as computed for all rays in a given projection was introduced as the Simultaneous Algebraic Reconstruction Technique (SART) [3]. To overcome the disadvantage that negative values appear in the reconstruction, another variant, the Multiplicative ART (MART) [4], can be used instead. Although MART algorithms produce less error at convergence compared to Additive ART, they converge only at small values of their relaxation parameter.

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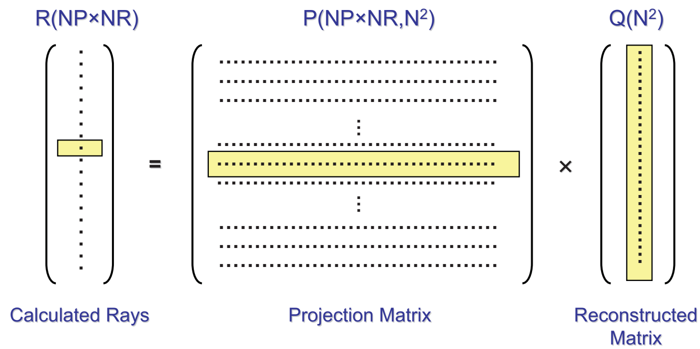

Fig. 1. Schematic representation for the intensity calculation of the ray R_i at a given angle from the Reconstructed Matrix Q_j with the help of the Projection Matrix P_{ij} according to Equation (1).

In electron microscopy, where limitations of the field of view and the contribution of scatter in the micrographs lead to relatively large deviations from the cosine behaviour of the tilt-angle, a Controlled Algebraic Reconstruction Technique (CART) [5] has been developed. This algorithm stabilizes the region of interest by dynamically scaling the input data during the procedure and is therefore able to operate in such a way that the central region, i.e., the object of interest is optimally reconstructed.

In the reconstruction procedure the intensity of each ray R_i at a given angle can be calculated from the Projection Matrix P_{ij} and the Reconstructed Matrix Q_j. The Projection Matrix P_{ij} is a weighting matrix, which carries the information of how much the j^{th} element of the matrix being reconstructed contributes to the i^{th}-ray. The Reconstructed Matrix Q_j represents the unknown pattern of the planar image to be reconstructed and has normally the shape of a square matrix with dimension $N \times N$, written in a vector (one-column) matrix, so that:

$$R_i = \sum_{j=1}^{N^2} P_{ij} \times Q_j \tag{1}$$

It is obvious that the index i runs in the interval $i = 1, 2, \ldots NP \times NR$, with NP the number of the projections (angles) where the ray intensity is measured and NR the number of the constant width rays per each projection, while $j = 1, 2, \ldots N^2$ (Fig. 1).

The main task of every reconstruction procedure in the computed tomography is to minimize the difference between the calculated R_i and the measured S_i ray intensity. In the following, the Algebraic Reconstruction Technique (ART), one of the most basic iterative techniques, will be discussed with respect to the minimization procedure based on the Newton-Raphson scheme. It will be shown that a simple definition of a *Cost Function* leads to the traditional form of the ART. Then, taking into account the information of neighborhood pixels, rays and projection angles, a new form of the *Cost Function* will be proposed. Based on this extended function, the improved version of the Newton-Raphson ART (NR-ART) will be presented. Convergence speed and quality of the reconstruction will be discussed in the case of simple software phantoms.

II. Newton-Raphson Scheme and ART

In numerical analysis, the Newton-Raphson method uses the iterative process

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \tag{2}$$

to approach the root x of a real-valued function with a quadratic convergence rate [6]. The method derives from the Taylor expansion of the function around a point and requires the evaluation of both, the function $f(x)$ and its derivative $f'(x)$. The same method can be used to locate extrema of a function, since the stationary points x for the function $f(x)$ are roots of the first derivative $f'(x)$. In this case, a similar iterative process applies

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)} \tag{3}$$

and the evaluation of both, the first and the second derivative of the twice-differentiable function, is required. In optimization problems this iterative scheme can be generalized to several dimensions by replacing the derivative with the gradient, $\nabla f(x)$, and the reciprocal of the second derivative with the inverse of the Hessian Matrix, $Hf(x)$:

$$x_{k+1} = x_k - [Hf(x_k)]^{-1} \nabla f(x_k) \tag{4}$$

Although there exist many methods to approximate the Hessian Matrix, there are cases where the Hessian is close to a non-invertible matrix and the whole process becomes numerically unstable.

The image reconstruction problem in the computed tomography is an optimization problem: We try to minimize the difference between the measured ray intensity S_i and its calculated value R_i by adjusting the elements Q_j of the matrix being reconstructed. In other words, we try to minimize a *Cost Function F* which is defined through the difference of measured-calculated ray at a given angle:

$$F(Q_j) = F(S_i - R_i) = F\left(S_i - \sum_{j=1}^{N^2} P_{ij} Q_j\right) \tag{5}$$

Therefore, the Newton-Raphson iterative process making use of Equation (3) is given by the relation:

$$Q_j^{k+1} = Q_j^k - \frac{F'(Q_j^k)}{F''(Q_j^k)} \tag{6}$$

Defining now the *Cost Function* as the quadratic difference between measured and calculated ray intensity

$$F(Q_j^k) = (S_i - R_i)^2 = \left(S_i - \sum_{j=1}^{N^2} P_{ij} Q_j^k\right)^2 \tag{7}$$

its first and second derivative can be evaluated as follows:

$$F'(Q_j^k) = \frac{\partial F}{\partial Q_j^k} = -2(S_i - R_i) P_{ij} \tag{8}$$

$$F''(Q_j^k) = \frac{\partial F'}{\partial Q_j^k} = 2 P_{ij} P_{ij} \tag{9}$$

By substituting these results to Equation (6) and taking care of the ray intensity normalization one obtains the relation

$$Q_j^{k+1} = Q_j^k + \frac{S_i - R_i^k}{\sum\limits_{j=1}^{N^2} P_{ij} P_{ij}} P_{ij} \tag{10}$$

which is the traditional algorithmic approach of the Algebraic Reconstruction Technique.

It is clear from the analysis above that the Newton-Raphson scheme with a *Cost Function* equal to the quadratic difference of the measured and calculated ray intensity leads exactly to the well known traditional ART.

III. The Extended Cost Function

A possible extension of the *Cost Function F* will be discussed in this section. The main idea is to include additional information in the function F, and consequently in the minimization procedure, besides the central quadratic difference $(S_i - R_i)^2$ and its derivatives with respect to the element Q_j under correction. As shown in the previous section, the basic mechanism of the traditional ART, which is based on this simple quadratic difference, lacks the ability to take into account corrections dictated by at least the neighborhood measured data. On the other hand, a heavily constructed *Cost Function* with a lot of additional information will be computationally non-efficient and it will probably lead to ART variants already discussed elsewhere.

Having in mind all the previous considerations, an obvious extension of the *Cost Function* has to include the most contiguous elements which show a major sensitivity with respect to the matrix element Q_j, namely all possible elements that absolutely maximize the gradient $\nabla F = \left(\frac{\partial F}{\partial Q_j}\right)_m$ for different directions m.

After a careful study of the problem, following three directions have been selected and included in the extended *Cost Function*:

- **Closest Pixels:** Apart from the central element Q_j under correction the most closets pixels to it, Q_{j-1} and Q_{j+1}, is also expected to affect the correction. Therefore they are added to the extended *Cost Function* and this extra term is then expressed by the sum:

$$(S_i - R_i)^2_{j-1} + (S_i - R_i)^2_{j+1}$$

- **Closest Rays:** It is natural to extend the information originated by the two neighbor rays R_{i-1} and R_{i+1} with respect to the pixel Q_j. The indexes are appropriately limited in the range of the given projection and the extra term in the *Cost Function* is formulated by the expression:

$$(S_{i-1} - R_{i-1})_j^2 + (S_{i+1} - R_{i+1})_j^2$$

- **Closest Projections:** For a sufficient large number of projections at different angles, it is expected that rays of the neighborhood projections drastically influence the reconstruction of the element Q_j. It is therefore fair to include in the *Cost Function* F rays from the previous and the next angle-projection. Since the number of rays per projection equals to NR, the corresponding ray in the previous and next projection is respectively R_{i-NR} and R_{i+NR} and the term added to F is:

$$(S_{i-NR} - R_{i-NR})_j^2 + (S_{i+NR} - R_{i+NR})_j^2$$

All these terms of the extended *Cost Function*, grouped in the three directional categories, are schematically represented in Fig. 2.

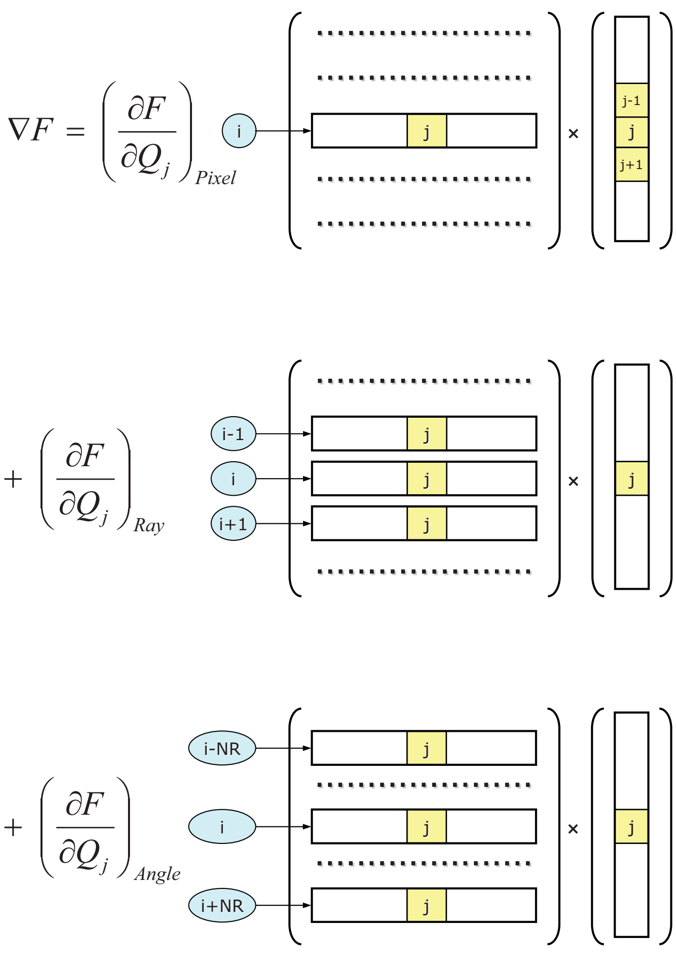

Fig. 2. Terms included in the extended *Cost Function* in a schematic representation. They can be grouped in three different categories, indicated as different directions in the gradient needed to construct the derivatives: (a) Contribution from the neighbor pixels (b) Contribution from the closest rays and (c) Contribution from the closest angle-projections. The exact expression of the *Cost Function* is given by the Equation (11).

Summarizing all the extra added contributions to the *Cost Function F*, its extended form can be now expressed as:

$$\begin{aligned} F(Q_j) = (S_i - R_i)_j^2 + (S_i - R_i)_{j-1}^2 + (S_i - R_i)_{j+1}^2 \\ + (S_{i-1} - R_{i-1})_j^2 + (S_{i+1} - R_{i+1})_j^2 \\ + (S_{i-NR} - R_{i-NR})_j^2 + (S_{i+NR} - R_{i+NR})_j^2 \end{aligned} \quad (11)$$

The first and the second derivative are calculated for each of the terms above with respect to the element Q_j of the matrix being reconstructed. For a given ray with index m it can be easily shown that:

$$\left(\frac{\partial F}{\partial Q_j}\right)_m = \frac{\partial}{\partial Q_j}(S_m - R_m)_j^2 = -2(S_m - R_m)P_{mj} \quad (12)$$

$$\left(\frac{\partial^2 F}{\partial Q_j^2}\right)_m = 2P_{mj}P_{mj} \quad (13)$$

Combining Equations (12) and (13) with the Newton-Raphson scheme of Equation (6), the individual correction after the k^{th} iteration step for each term of the extended *Cost Function* is:

$$X_{mj} = \frac{S_m - R_m^k}{\sum_{j=1}^{N^2} P_{mj}P_{mj}} P_{mj} \quad (14)$$

so that the totaly accumulated correction by all terms can be expressed as:

$$Q_j^{k+1} = Q_j^k + \frac{1}{C}\sum_{mj} X_{mj} \quad (15)$$

In the sum above and for the proposed *Cost Function* there are altogether seven terms with:

$$\begin{aligned} (m,j) \in \{(i,j-1),(i,j),(i,j+1)\} \\ \cup\{(i-1,j),(i+1,j)\} \\ \cup\{(i-NR,j),(i+NR,j)\} \end{aligned} \quad (16)$$

The factor C plays the role of a normalization factor and is directly dependent on the number of terms in the summation.

IV. Algorithmic Implementation

The Newton-Raphson correction approach with the extended *Cost Function* as described in the previous section has been implemented within the traditional iterative ART scheme. The new algorithm, referred in the following as *Newton-Raphson Algebraic Reconstruction Technique (NR-ART)*, has been tested with several software phantoms. Each group of the contributing terms included in the extended *Cost Function*, which is interpreted as a different direction in the gradient, can be separately activated in the reconstruction procedure. It is possible, therefore, to study quantitatively the partial effect of these terms in the final results.

Software test patterns have been constructed in a square matrix format with dimensions 64x64 and 128x128 and their projections have been created for equidistant angle intervals in the range $(0^0 \ldots 180^0)$. In Fig. 3 the convergence of the reconstruction procedure for a matrix with N=64 and for two different sets, with 18 and 36 projections respectively, is

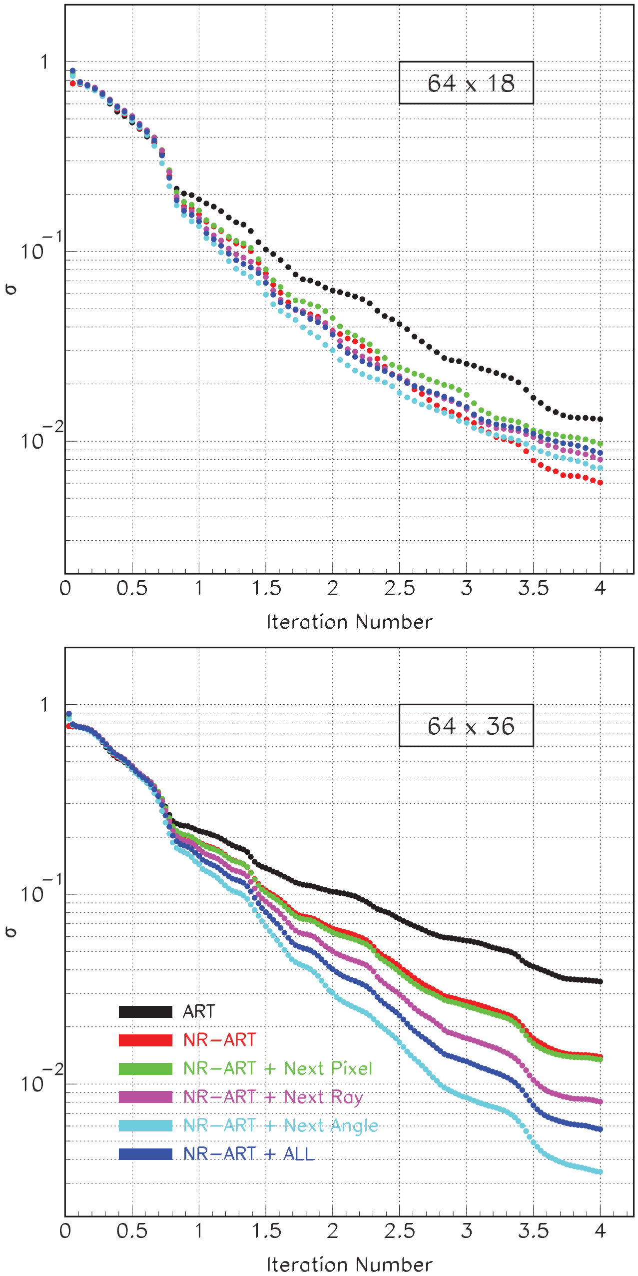

Fig. 3. Convergence of the reconstruction procedure for a square matrix phantom with N=64 for two different sets with 18 and 36 projections respectively. The traditional ART scheme is directly compared with the NR-ART scheme based on the here proposed extended *Cost Function*. The evolution of the different contributing groups are separately plotted and indicated with different color. The definition of the mean quadratic error σ is given in Equation (17).

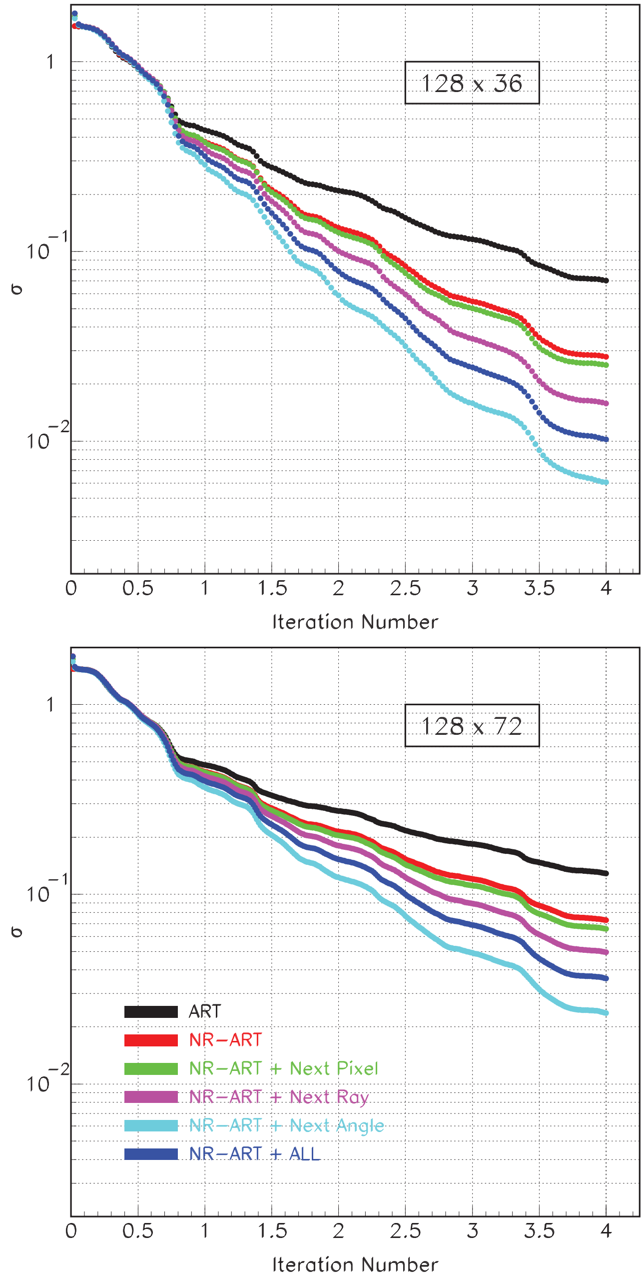

Fig. 4. Convergence of the reconstruction procedure for a square matrix phantom with N=128 for two different sets with 36 and 72 projections respectively. Details are the same as in the previous Fig. 3.

shown. The evolution of the mean quadratic error σ, defined as:

$$\sigma = \sqrt{\frac{\sum^{rr}(S_i - R_i)^2}{rr}} \quad with \quad rr = NP \times NR \qquad (17)$$

with the number of grand iterations in the reconstruction procedure is plotted.

According to the plotted results for the three different contributing groups defined in the extended *Cost Function* following conclusion can be drawn:

- For small dimensions N of the Reconstructed Matrix and for a small Number of Projections NP, although the NR-ART scheme with only the central term shows improved results, the terms in the extended *Cost Function* don't contribute significantly in the quality of the image reconstruction.
- For a sufficient large Number of Projections NP, each of the three groups in the extended function shows a significant contribution at least after the first iteration. The best results are obtained for the closest angle-projections group (NR-ART + Next Angle) followed by the closest

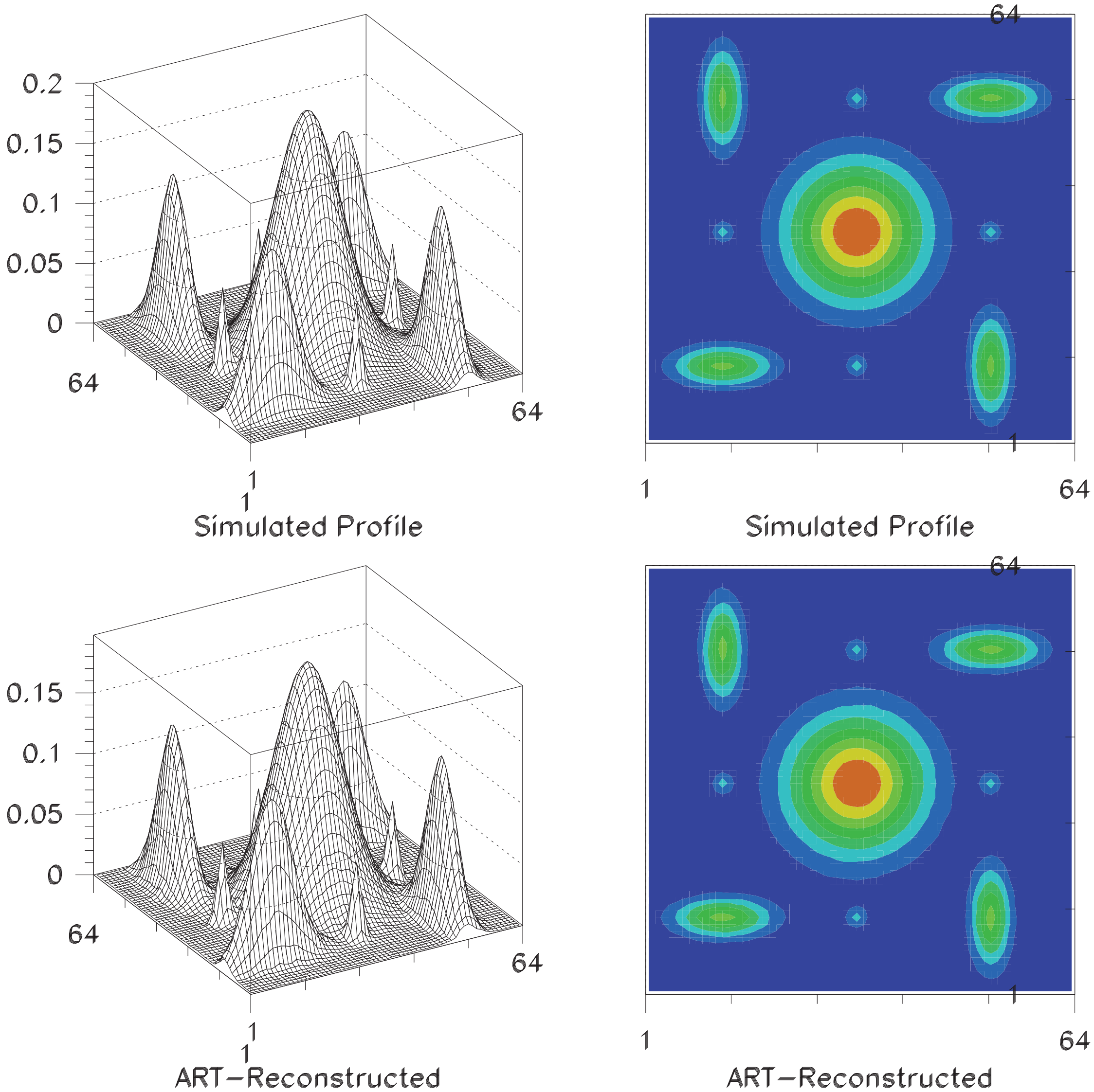

Fig. 5. NR-ART reconstruction of a software pattern consisting of 9 Gauss shaped ellipsoids with matrix dimension 64x64 and NP=36 projections covering an equidistant angle range from 0^0 to 180^0. The 3-dimensional pattern and its equivalent contour plot for the generated and the reconstructed image are shown together for direct comparison.

rays group (NR-ART + Next Ray). The contribution of the closest pixels (NR-ART + Next Pixel) is minimal and it will be improved only in higher matrix dimensions, as shown in Fig. 4.

- Activation of all correcting groups (NR-ART + ALL) shows in general worse results than the closest angle-projection group alone.

Similarly, in Fig. 4 the convergence of the reconstruction procedure for a matrix with N=128 and for two different sets, with 36 and 72 projections respectively, is shown.

Qualitative results of the here analyzed NR-ART scheme are shown in Fig. 5. A software pattern created in a square matrix form with N=64 and consisting of 9 Gauss shaped ellipsoid of different sizes has been reconstructed from the 36 projections covering an equidistant angle range from 0^0 to 180^0. The 3-dimensional pattern and its equivalent contour plot for the generated and the reconstructed image are shown for direct comparison in the same figure. Because of the simplicity of the generated phantom and its symmetry, the quality of the reconstruction is ideal and no optical differences are identifiable.

Finally, the well known Shepp-Logan head phantom, which consists of a number of ellipses of varying sizes and densities, has been also reconstructed using the NR-ART approach. The resulted image is shown in Fig. 6 together with the original and the image reconstructed with the traditional ART algorithm. The matrix size in this example was 128x128; again a number of 36 equidistant projections in the angle range $(0^0 \ldots 180^0)$ have been used in the reconstruction. The reconstruction convergence for many other, more complicated, software phantoms has been studied in the context of the here described methodology [7].

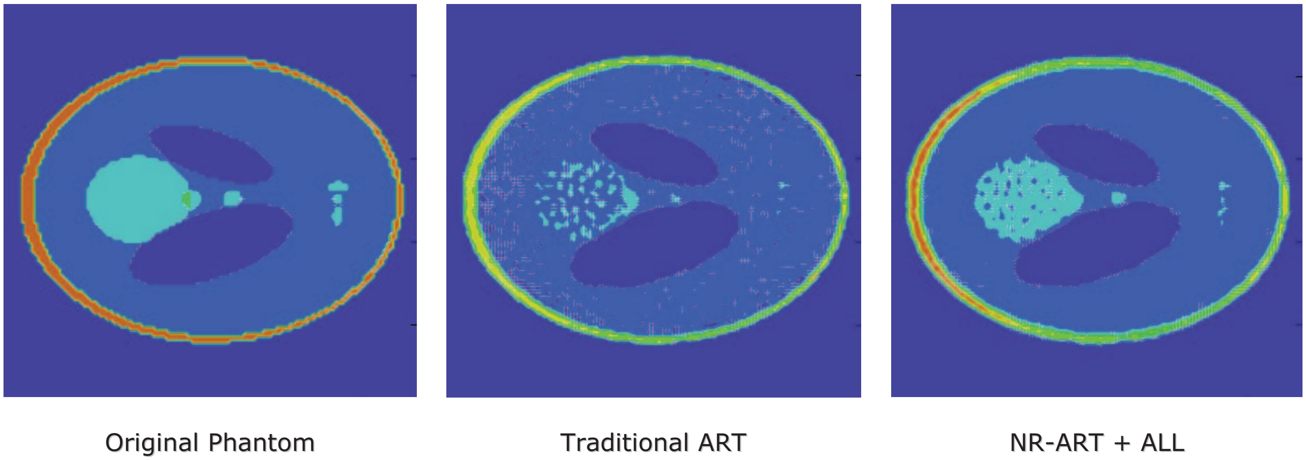

Fig. 6. Reconstruction of the Shepp-Logan head phantom. Left is the original phantom, in the center the reconstructed with the traditional ART and right the reconstructed with the NR-ART approach presented in this work. Matrix dimension is 128x128; 36 equidistant projections in the angle range $(0^0 \dots 180^0)$ have been used in the reconstruction.

All tests have been performed on an 8-core Unix platform with Xeon CPU running at 2.50 GHz. For the implementation of the reconstruction algorithm the GNU-Compiler 2.6.9, ELF-64 bit has been used. Measured CPU times for a grand iteration on this platform are shown in the following table.

N x NP	CPU-Time per Iteration
64 x 18	0.16 s
64 x 36	0.31 s
128 x 36	2.53 s
128 x 72	6.81 s

V. Concluding Remarks

In this work, the image reconstruction problem has been discussed in the context of the optimization procedure. It is shown that the Newton-Raphson algorithmic approach for finding the stationary point of a *Cost Function*, which in its simplest definition consists of the quadratic difference of measured-calculated ray at a given angle, directly leads to the traditional Algebraic Reconstruction Technique (ART). Based on this fact, an extended *Cost Function* has been defined in an economical way, taking into account all the possible information from the closest pixels, rays and angle-projections of the matrix element being reconstructed. This Newton-Raphson ART scheme shows a significant improvement in the convergence allowing a speedup of the whole reconstruction procedure. The effect to the final reconstruction result for each term of the extended *Cost Function* has been quantitatively studied with simple software phantoms.

VI. Acknowledgement

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